MA1 - zkouška 22.1.21 - dopoledne

(1) $$\int \left(\frac{1}{x^3} \ln\left(1-\frac{1}{x^2}\right) + \frac{7e^{2x}-4}{e^{2x}-2e^x+2} \right) dx = I_1 + I_2$$

a) existence - max. interval, kde integrand je spojitá funkce:

(i) $\frac{7e^{2x}-4}{e^{2x}-2e^x+2}$ je def. v $\mathbb{R}$, spojitá fce v $\mathbb{R}$

$\ln\left(1-\frac{1}{x^2}\right)$ je def. pro $x \neq 0$ a takové, že $1-\frac{1}{x^2} > 0$, tj. $\frac{x^2-1}{x^2} > 0$, tj. $x \in (-\infty,-1)$ nebo $x \in (1,+\infty)$ — interval 0,5

(ii) $$I_1 = \int \frac{1}{x^3} \ln\left(1-\frac{1}{x^2}\right) dx \underset{1VS}{=} \left| \begin{array}{l} 1-\frac{1}{x^2} = t \\ +\frac{2}{x^3}dx = dt \end{array} \right| = \frac{1}{2} \int \ln t \, dt \underset{PP}{=}$$

2b substituce

1b $$\underset{PP}{=} \left| \begin{array}{l} u' = 1, u = t \\ v = \ln t, v' = \frac{1}{t} \end{array} \right| = \frac{1}{2}\left(t\ln t - \int t \cdot \frac{1}{t} dt\right) = \frac{1}{2}\left(t \ln t - t\right) + C =$$

$$= \frac{1}{2}\left(1-\frac{1}{x^2}\right)\left(\ln\left(1-\frac{1}{x^2}\right)-1\right) + C \, ;$$

(iii) $$\int \frac{7e^{2x}-4}{e^{2x}-2e^x+2} dx \underset{2VS}{=} \left| \begin{array}{l} e^x = t \\ x = \ln t \\ dx = \frac{1}{t}dt \end{array} \right| = \int \frac{7t^2-4}{(t^2-2t+2)t} dt =$$

2b substituce

$$= -2\int \frac{1}{t}dt + \int \frac{9t-4}{t^2-2t+2}dt = -2\int \frac{1}{t}dt + \frac{9}{2}\int \frac{2t-2}{t^2-2t+2}dt + 5\int \frac{1}{(t-1)^2+1}dt$$

$$= -2\ln|t| + \frac{9}{2}\ln(t^2-2t+2) + 5\,\mathrm{arctg}(t-1) + C =$$

$$= -2x + \frac{9}{2}\ln\left(e^{2x}-2e^x+2\right) + 5\,\mathrm{arctg}(e^x-1) + C$$

$(= -2\ln e^x)$ (0,5b) 2b — 2,5 integrace

Rozklad: $$\frac{7t^2-4}{t(t^2-2t+2)} = \frac{A}{t} + \frac{Bt+C}{t^2-2t+2}, \text{ tj.}$$

$$7t^2-4 = A(t^2-2t+2) + Bt^2 + Ct, \text{ tj.}$$ 2b rozklad

u t^2: $A + B = 7 \Rightarrow \underline{B = 9}$

u t: $-2A + C = 0 \Rightarrow \underline{C = -4}$

u t^0: $2A = -4 \Rightarrow \underline{A = -2}$

(2) $f(x) = \operatorname{arctg}\left(\frac{2x}{1-x^2}\right)$

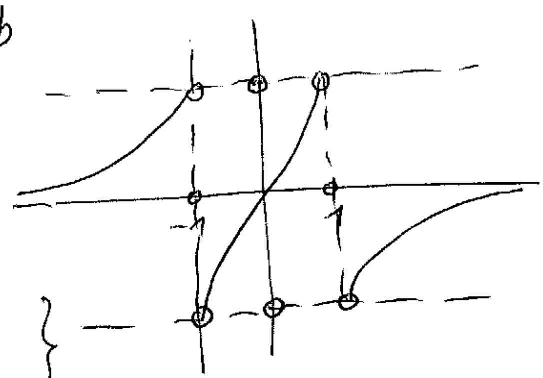
a) $D_f = R \setminus \{\pm 1\} = (-\infty,-1) \cup (-1,1) \cup (1,+\infty)$.
f je spojitá v D_f; f je lichá v D_f
$f(x)=0 \Leftrightarrow x=0$, $f(x)>0$ v $(0,1) \cup (-\infty,-1)$
$f(x)<0$ v $(-1,0) \cup (1,+\infty)$ } 1b

$\lim\limits_{x\to\pm\infty} \operatorname{arctg}\left(\frac{2x}{1-x^2}\right) = \lim\limits_{\substack{\text{VLSF} \\ y\to 0}} \operatorname{arctg} y = 0$ 1b

$\lim\limits_{x\to 1\pm} \operatorname{arctg}\left(\frac{2x}{1-x^2}\right) = \lim\limits_{y\to\mp\infty} \operatorname{arctg} y = \mp\frac{\pi}{2}$ 1b }

$H_f \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, analog. $\lim\limits_{x\to -1\pm} \operatorname{arctg}\left(\frac{2x}{1-x^2}\right) = \mp\frac{\pi}{2}$

odhad grafu:



b) $f'(x) = \dfrac{1}{1+\frac{4x^2}{(1-x^2)^2}} \cdot \dfrac{2\left[(1-x^2) - x\cdot(-2x)\right]}{(1-x^2)^2} = 2\dfrac{1-x^2+2x^2}{1-2x^2+x^4+4x^2} =$

$= 2\dfrac{1+x^2}{(1+x^2)^2} = \dfrac{2}{1+x^2}$, $f'(x)>0 \Rightarrow$ [illegible] ↗ 1b

f' 1b

$\Rightarrow$ f je rostoucí v $(-\infty,-1)$, v $(-1,1)$, v $(1,+\infty)$, f nemá lok. extrémy, ani globální (je def. v otevřených intervalech, max, resp. min. jsou limity) nemá extrémy 1b

f'' 1b

c) $f''(x) = \dfrac{2\cdot(-2x)}{(1+x^2)^2}$, $f''(x)=0 \Leftrightarrow x=0$

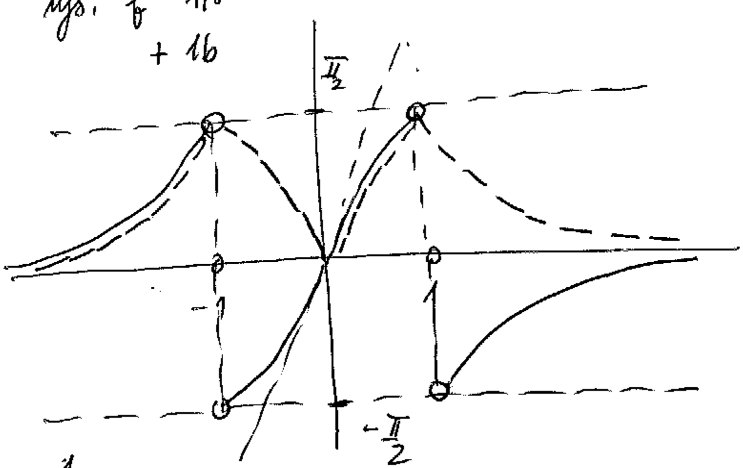
vyš. f'' 1,5
+ 1b

f''	+		+		−		−
	∪	-1	∪	0	∩	1	∩

$\Rightarrow$ v bodě $x=0$ má f inflexi,
$f'(0) = 2$

d) asymptota grafu: osa x,
tj. $y=0$

graf $|f|$ ----

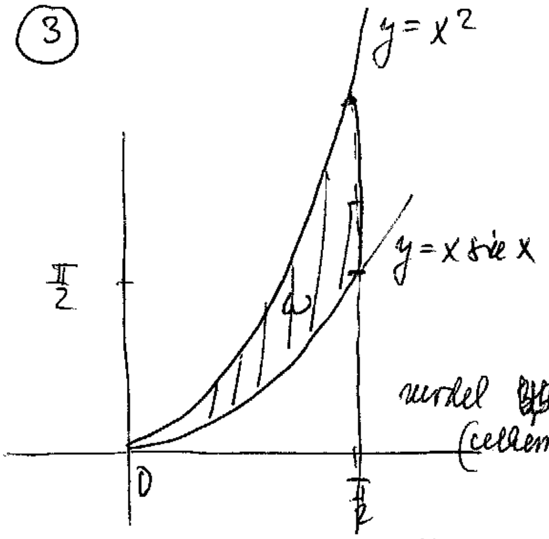


$\lim\limits_{x\to 1+} f'(x) = \lim\limits_{x\to 1-} f'(x) = \lim\limits_{x\to 1} \dfrac{2}{1+x^2} = 1$

asymptota 0,5b

graf f 1b
$|f|$ 0,5b

③



$S(\omega) = \int_0^{\frac{\pi}{2}} (x^2 - x\sin x)\,dx$, ~~b~~ 2

nebot: 1) $x^2 = \sin x \cdot x$ pro $x=0$

mez 0,5b 2) pro $x>0$ je

$x > \sin x$, tj. $(x>0)$

nerdel ~~b~~ 4b (celkem) 1,5 $x^2 > x\sin x$ pro $x>0$

Vypočet $S(\omega)$: 3b celkem

$$\int_0^{\frac{\pi}{2}} (x^2 - x\sin x)\,dx = \int_0^{\frac{\pi}{2}} x^2 dx - \int_0^{\frac{\pi}{2}} x\sin x\,dx =$$

(0,5b) ~~(b)~~ 2,5b

$$= \left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}} - 1 = \frac{1}{3}\frac{\pi^3}{8} - 1 = \frac{\pi^3}{24} - 1 \quad (>0)$$

$$\int_0^{\frac{\pi}{2}} x\sin x\,dx = \left| \begin{matrix} u' = \sin x, & u = -\cos x \\ v = x, & v' = 1 \end{matrix} \right| = \left[-\cos x \cdot x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,dx =$$

2,0b

$$= \underbrace{\left[-x\cos x\right]_0^{\frac{\pi}{2}}}_{=0} + \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$$ dosazení 0,5b

④ $y' = \frac{2}{1+x}(1-y)$ $x \in (-\infty,-1)$ nebo $x \in (-1,+\infty)$ $(= \mathcal{I}_2)$
$y \in \mathbb{R}$ $(= \mathcal{I}_1)$

a) (i) $y(x)=1$, $x\in(-\infty,-1)$ nebo $x\in(-1,+\infty)$ – stac. řešení 1b

(ii) $y(x)\neq 1$ pro $x\in(-\infty,-1)$ i $x\in(-1,+\infty)$, pak separace:

$$\int \frac{dy}{1-y} = \int \frac{2}{1+x}\,dx,$$ tj. separace 1b (bez $y\neq 1$ 0,5b)

$$-\ln|y-1| = 2\ln|1+x| + C$$ integrace 2b

tj. $\ln|y-1| = \ln(1+x)^{-2} + C$

$|y-1| = e^C \cdot \frac{1}{(1+x)^2}$, úpravy 1,5b

$y-1 = \frac{e^C}{(1+x)^2}$ pro $y>1$, $x\in\mathcal{I}_1$, $x\in\mathcal{I}_2$

$y-1 = \frac{-e^C}{(1+x)^2}$ pro $y<1$ –"– 1,5b

tj. $y(x) = 1 + \frac{k}{(1+x)^2}$, $k\neq 0$, $x\in\mathcal{I}_1, x\in\mathcal{I}_2$

a (i) a (ii): $y_{ob} = 1 + \frac{k}{(1+x)^2}$, $k\in\mathbb{R}$, $x\in\mathcal{I}_1, x\in\mathcal{I}_2$.

b) počáteční úlohy:

(i) $y(0)=2$: $2 = 1+k \Rightarrow k=1$ 0,5b

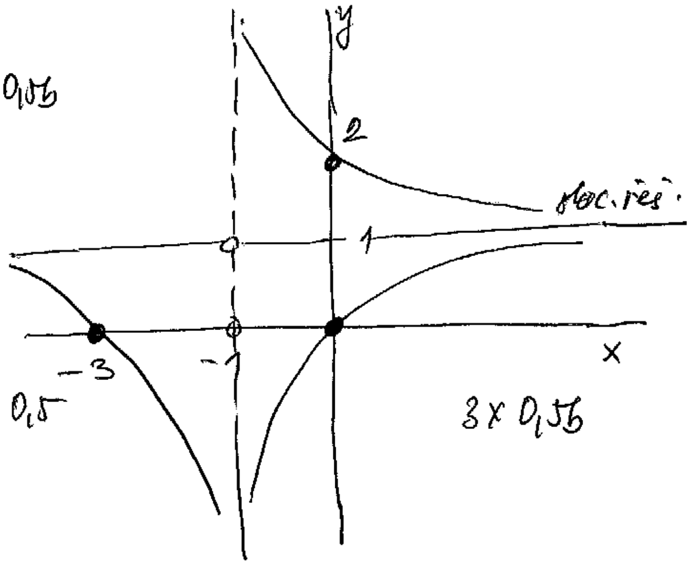
a $y(x) = 1 + \frac{1}{(1+x)^2}$, $x\in(-1,+\infty)$

(ii) $y(0)=0$: $0 = 1+k \Rightarrow k=-1$

a $y(x) = 1 - \frac{1}{(1+x)^2}$, $x\in(-1,+\infty)$ 0,5

(iii) $y(-3)=0$: $0 = 1 + \frac{k}{4} \Rightarrow k=-4$

$y(x) = 1 - \frac{4}{(1+x)^2}$, $x\in(-\infty,-1)$ 0,5b



$$\lim_{x\to -1\pm} y(x) = \lim_{x\to -1\pm}\left(1+\frac{k}{(1+x)^2}\right) = \begin{cases} +\infty, & k>0 \\ -\infty, & k<0 \end{cases}$$

$$\lim_{x\to \pm\infty} y(x) = \lim_{x\to\pm\infty}\left(1+\frac{k}{(1+x)^2}\right) = 1$$

Otázky „teoretické"

① (i) $B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$ B je regulární matice ($h(B) = 3$), tedy existuje k B matice inverzní B^{-1},

tj. matice, pro kterou platí $B \cdot B^{-1} = B^{-1}B = I$ (I - jednotková matice) 1b

výpočet B^{-1}:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right) \Rightarrow B^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}$$

ex. B^{-1} 1b
3b výpočet

(ii) „zkouška"; tj. $\overset{?}{}$ $B \cdot B^{-1} = I$:

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix}$$

zkouška 2b

Řešení soustavy: $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, tj. $B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$,

pak $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = B^{-1}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, tj. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$ 2b

(tedy $B . X = b \;|\cdot B^{-1}$,

$B^{-1}(BX) = B^{-1} \cdot b$ a

$(B^{-1} \cdot B) . X = B^{-1}b$, tj.

$\underbrace{I}\cdot \underline{X = B^{-1}b}$)

(2) b) $f(x) = \dfrac{\ln(1+x^2)}{x}$, $x \neq 0$; $f(0) = 0$ –

(i) ? f je spojitá v bodě $x_0 = 0$ $\Longleftrightarrow$ $\lim\limits_{x\to 0} f(x) = 0$ $\Longleftrightarrow$ $\lim\limits_{x\to 0} \dfrac{\ln(1+x^2)}{x} = 0$?

„výpočet": $\lim\limits_{x\to 0} \dfrac{\ln(1+x^2)}{x} = „\dfrac{0}{0}" \stackrel{l'H}{=} \lim\limits_{x\to 0} \dfrac{\frac{1}{1+x^2}\cdot 2x}{1} = „\dfrac{0}{1}" \stackrel{=}{AL} 0$

1b „formulace" 3b výpočet

$\Rightarrow$ f je spojitá v bodě $x_0 = 0$.

(ii) $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x} = \lim\limits_{x\to 0} \dfrac{\ln(1+x^2)}{x^2} = „\dfrac{0}{0}" \stackrel{l'H}{=} \lim\limits_{x\to 0} \dfrac{\frac{1}{1+x^2}\cdot \cancel{2x}}{\cancel{2x}} = 1$

1b „formulace" 3b výpočet

Limity lze též počítat užitím (T) $\lim\limits_{h\to 0} \dfrac{\ln(1+h)}{h} = 1$; pak

$\lim\limits_{x\to 0} \dfrac{\ln(1+x^2)}{x} = \lim\limits_{x\to 0} \underbrace{\dfrac{\ln(1+x^2)}{x^2}}_{\to 1} \cdot x \underset{AL}{=} 0$, $\lim\limits_{x\to 0} \dfrac{\ln(1+x^2)}{x^2} \underset{T, VLSF}{=} 1$

(3) $f(x) = \cos(e^{4x}-1)$ $T_2^{f,0}(x) = f(0) + f'(0)\cdot x + \dfrac{f''(0)}{2} x^2$:

$f(0) = \cos 0 = 1$

$f'(0) = -\sin(e^{4x}-1)\cdot e^{4x}\cdot 4\Big|_{x=0} = 0$ 1,5b

$f''(0) = -\cos(e^{4x}-1)(e^{4x}\cdot 4)^2 - \sin(e^{4x}-1)\cdot e^{4x}\cdot 4^2\Big|_{x=0} = -16$, 1,5b

tj. $T_2(x) = 1 - \frac{16}{2}\cdot x^2$, tj. $T_2(x) = 1 - 8x^2$ 3b